Δelta 2026

Day 1 - Queueing Theory

Congestion and Bottlenecks

Time: 3 hours • Total: 50 points

Each problem is scored independently. Justify all answers with complete proofs. Partial credit is awarded for significant progress. No calculators or electronic devices are permitted.

Problem 1 — 5 points

The Bridge of Constant Flow. A new toll bridge connects two islands. Vehicles arrive at the entrance of the bridge according to a Poisson process with a constant rate of λ vehicles per minute. The time it takes for any vehicle to cross the bridge depends on its individual speed, which is an independent random variable such that the expected time to completely cross the bridge is W minutes.

Assuming the traffic flow has been running infinitely long and is in a steady state, prove analytically that the expected number of vehicles physically on the bridge at any given moment is exactly λW.

Problem 2 — 8 points

The Sequential Scanner Invariance. An automated traffic control system consists of N vehicle scanners arranged sequentially on a strict single-lane road. The j-th scanner takes exactly a deterministic time of $s_j > 0$ minutes to scan any car.

A sequence of K cars enters the road. They arrive at the very first scanner at recorded times $t_1 \leq t_2 \leq \cdots \leq t_K$. Cars strictly cannot pass one another. If a car arrives at a scanner that is currently busy processing the car ahead of it, it waits in a queue.

Let $T(s_1, s_2, \ldots, s_N)$ be the exact timestamp when the final (K-th) car completely finishes its scan at the final (N-th) scanner.

Prove that T is perfectly invariant under any rearrangement (permutation) of the scanner processing times $(s_1, s_2, \ldots, s_N)$.

Problem 3 — 10 points

The Inspection Paradox. A city transit authority operates a shuttle service where the time intervals between consecutive shuttle arrivals at a central station are independent and identically distributed random variables with mean $\mu > 0$ and variance $\sigma^2 > 0$.

A city inspector arrives at the central station at a completely random time (independent of the shuttle schedule) to evaluate the service.

Prove that the expected time the inspector must wait until the next shuttle arrives is strictly greater than $\frac{\mu}{2}$. Specifically, compute this expected waiting time in terms of μ and σ^2.

Problem 4 — 12 points

The Roundabout Network. A city's downtown grid consists of K interconnected roundabouts. Vehicles enter the network from the suburbs into roundabout i according to a Poisson process with rate γ_i (for $i = 1, \ldots, K$). Navigating roundabout i takes an exponentially distributed amount of

time with mean $1/\mu_i$. Upon exiting roundabout i, a vehicle proceeds to roundabout j with a fixed probability P_{ij}, or exits the city downtown entirely with probability $1 - \sum_{j=1}^{K} P_{ij}$.

Let $(\lambda_1, \ldots, \lambda_K)$ be the unique solution to the traffic flow equations

$$\lambda_i = \gamma_i + \sum_{j=1}^{K} \lambda_j P_{ji},$$

ensuring $\lambda_i < \mu_i$ for all i. Let N_i be the number of cars currently at roundabout i.

Prove analytically that in the steady state, the joint probability $P(N_1 = n_1, N_2 = n_2, \ldots, N_K = n_K)$ completely factorizes into $\prod_{i=1}^{K} P(N_i = n_i)$, and find an explicit formula for $P(N_i = n_i)$.

Problem 5 — 15 points

The Heavy Traffic Bottleneck. A highway narrows abruptly, creating a highly congested bottleneck. The inter-arrival times of vehicles at the bottleneck are independent and identically distributed (i.i.d.) random variables with mean $1/\lambda$ and variance σ_a^2. The time it takes a vehicle to pass the bottleneck is also i.i.d. with mean $1/\mu$ and variance σ_s^2. The traffic intensity $\rho = \lambda/\mu$ is strictly less than 1, preventing infinite traffic jams, but the system operates in "heavy traffic" as $\rho \to 1^-$.

Let W be the steady-state expected waiting time for a car before it can enter the bottleneck. Prove analytically that as $\rho \to 1^-$, the expected waiting time is asymptotically given by:

$$W \approx \frac{\lambda\left(\sigma_a^2 + \sigma_s^2\right)}{2(1-\rho)}.$$

End of Round 2