Δelta 2026

Day 1 - Graph Theory

Adaptive Flows & Grid Congestion

Time: 3 hours • Total: 50 points

Each problem is scored independently. Justify all answers with complete proofs. Partial credit is awarded for significant progress. No calculators or electronic devices are permitted.

Problem 1 — 5 points

A city has a road network consisting of n intersections (vertices) and some two-way roads (edges) connecting them. The traffic planning department assigns a "traffic intensity factor" to each intersection to model congestion. Let A be the $n \times n$ adjacency matrix of the road network, where $A_{ij} = 1$ if there is a road between intersection i and intersection j, and 0 otherwise.

A vector $x \in \mathbb{R}^n$ (where $x \neq 0$) of traffic intensity factors is considered a **steady state of congestion** with a multiplier λ if $Ax = \lambda x$. Let Δ be the maximum number of roads meeting at any single intersection.

Prove that the absolute value of any steady-state traffic multiplier λ is at most Δ.

Problem 2 — 8 points

To prevent continuous "block-circling" traffic jams, a city developer ensures that in their undirected two-way road network of n intersections, there are absolutely no 4-cycles (i.e., no set of four intersections directly connected in a loop; the graph is C_4-free). To keep the city highly connected despite this restriction, the developer builds as many bidirectional roads as possible.

Prove that the maximum possible number of roads they can build is bounded above by

$$\frac{n}{4}\left(1 + \sqrt{4n - 3}\right).$$

Problem 3 — 10 points

A city's two-way road network is represented by a graph G with n intersections. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be all the eigenvalues of the adjacency matrix A of this road network.

(a) Prove that the total number of round trips of length k (a sequence of k roads starting and ending at the same intersection, where roads and intersections can be repeated) across the entire city is exactly

$$\sum_{i=1}^{n} \lambda_i^k.$$

(b) The city is divided into a "Residential Zone" and a "Commercial Zone." The zoning laws are such that every road in the city only connects an intersection in the Residential Zone to an intersection in the Commercial Zone. Prove that for any odd integer k,

$$\sum_{i=1}^{n} \lambda_i^k = 0.$$

Problem 4 — 12 points

A futuristic city contains n main traffic hubs. For peak efficiency, between every single pair of hubs, there exists exactly one bidirectional hyperlane (meaning the network is a complete graph K_n).

The traffic controller operates using *Inter-District Green Waves*. In a single green wave k, the hubs are divided into two disjoint sets, X_k and Y_k, and exactly all hyperlanes connecting a hub in X_k to a hub in Y_k are given a green light.

To avoid accidents, a routing protocol demands that over the course of a day, every single hyperlane in the city must be given a green light during *exactly one* green wave. No hyperlane can be opened twice.

Prove that to satisfy this protocol, the traffic controller must schedule at least $n - 1$ distinct green waves.

Problem 5 — 15 points

A central AI is constructing the ultimate symmetric transit matrix for N identical stations. To prevent bottlenecks, the blueprint dictates that:

(i) Every station must be directly connected to exactly D other stations by bidirectional expressways.

(ii) The network contains no triangular loops (no three stations are mutually connected to each other).

(iii) If two stations are **not** directly connected by an expressway, there must exist exactly **one** unique sequence of 2 expressways connecting them (acting as an optimal 2-step commute).

By establishing a structural constraint on the transit matrix eigenvalues, prove that the degree D cannot be chosen arbitrarily, and determine all possible values that D can take.

End of Round 1